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NONLINEAR STABILITY OF AXISYMMETRIC SWIRLING FLOWS

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CONTENTS

We establish sufficient conditions for the nonlinear stability of incompressible, inviscid, swirling flows using the Arnol'd energy-Casimir method. We derive an axisymmetric Lie–Poisson bracket and work with equations of motion in swirl-function–vortex-density form. The flows and perturbations we consider may have axial variations. The formulation is closely analogous to that of two-dimensional, stratified, Boussinesq flows considered by Abarbanel *et al.* (*Phil. Trans. R. Soc. Lond.* A **318**, 349–409 (1986)) and a high-wavenumber cut off is necessary to overcome indefiniteness, as in that case. We give several examples of columnar swirling flows and discuss the relation of our results to linear stability studies of swirling flows.

1. Introduction

The stability of inviscid, axisymmetric, swirling flows is an important subject because of numerous problems of meteorological and aerodynamic significance which are well approximated by such flows. Moreover, the rich phenomena of vortex breakdown have recently prompted renewed interest in this area. See Leibovich (1978, 1984) for background. In the present paper we establish sufficient conditions for the nonlinear stability of such swirling flows to finite axisymmetric disturbances.

In linear stability studies of inviscid flows one seeks conditions on the 'base' flow that ensure that none of the eigenvalues of the linearized operator describing the evolution of perturbations have positive real part (cf. Joseph 1976; Drazin & Reid 1981). The stability of swirling flows to axisymmetric infinitesimal perturbations was first considered by Rayleigh (1880, 1916), who found that a 'pure' vortex with velocity field $(u_r, u_\theta, u_z) = (0, V(r), 0)$, in cylindrical coordinates, is stable only if $\Phi \equiv r^{-3}\,\mathrm{d}/\mathrm{d}r(\kappa^2) > 0$ for all r in the domain of interest ($(0, R)$ or $(0, \infty)$). Here $\kappa(r) = rV(r)$ is the swirl function. Subsequently, Synge (1933) showed that $\Phi > 0$ is necessary and sufficient for stability. Most recently, Howard & Gupta (1962) derived a sufficient condition for linear stability of a vortex with velocity $(0, V(r), W(r))$, including an axial component. The condition is

$$J \equiv \Phi \bigg/ \left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 > \frac{1}{4}, \tag{1.1}$$

where $\Phi = r^{-3}\,\mathrm{d}/\mathrm{d}r(r^2V^2(r))$ as before. J is a Richardson number and must exceed $\frac{1}{4}$ everywhere in the domain of interest. Note that all these results concern columnar flows: flows having no z-dependence.

Non-axisymmetric perturbations of columnar vortices have also been treated by the method of linear stability analysis. Non-axisymmetric modes seem generally to be more liable to instability than axisymmetric modes. Necessary conditions for linear instability were given by Rayleigh (1880) and Fjortoft (1950). Maslowe & Stewartson (1982) and Leibovich & Stewartson (1983) have recently given sufficient conditions for instability. For a review of the current status of inviscid, linear stability studies of swirling flows, see Leibovich (1984).

It is well known that linear stability does not imply nonlinear stability for conservative problems, even in finite dimensional systems (Holm *et al.* 1985). Nonlinear stability of an equilibrium solution requires that small but finite perturbations remain uniformly bounded for all time as they evolve under the full nonlinear equations. Linear stability is neither necessary, nor sufficient, for this. The method we use is an extension of the classical notion of Liapunov functions; it depends upon finding a constant of motion that has a local maximum or minimum at the equilibrium point under study. Arnol'd (1965, 1969) was the first to apply the method to two (space)-dimensional, inviscid fluid systems. Taking the kinetic energy plus conserved quantities that correspond to symmetries of the system via Noether's theorem, he formed an 'Arnol'd' function, which is a constant of the motion, and proved nonlinear stability through a convexity analysis of this function near the equilibrium in question. The theory has subsequently been generalized and applied to a wide range of solid, fluid and plasma systems. The reader should consult Holm *et al.* (1985) for more background information and general discussions of the method. Benjamin (1976) has used similar arguments, involving the energy functional and flow force, in connection with the stability of vortex rings. Also see Wan & Pulvirente (1985) and Wan (1988) for a complete treatment of vortex patches and axisymmetric vortex rings and Hill's spherical vortex, and see Benjamin (1984) for general background on variational methods in fluids.

The vortex breakdown phenomenon involves predominantly axisymmetric disturbances, although non-axisymmetric modes do play a significant role (Leibovich 1984). It is therefore appropriate to treat nonlinear stability of axisymmetric, swirling flows to axisymmetric perturbations. This makes the flows 'essentially' two-dimensional, and avoids what seem to be insuperable problems in the application of the Arnol'd method to fully general three-dimensional flows (Holm *et al.* 1985). Because vortex breakdowns of the bubble type (Leibovich 1978) are spatially localized, we shall restrict ourselves to perturbations occupying a finite, cylindrical spatial domain: one may think of the test section of an experimental flow apparatus.

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The paper is arranged as follows. In §2 we express the inviscid, axisymmetric Euler equations as evolution equations for swirl function, κ, and azimuthal vortex density, χ. Henceforth we work in these variables, along with the Stokes stream function ψ from which χ can be derived. We derive an axisymmetric Lie–Poisson bracket with respect to which the evolution equations are hamiltonian and we discuss various constants of motion, most of them related to classical hydrodynamic quantities. Section 3 contains the major analysis of the Arnol'd function A and the formulation of abstract criteria for nonlinear stability. In §4 we address the fundamental problem of indefiniteness of the second variation of A and solve it with a high-wavenumber cut off. Sections 5, 6 and 7 contain discussions of special classes of flows: columnar, non-rotating and flows in annular domains, for which the general theory must be modified. Examples of several columnar flows are given in §8 and explicit stability criteria are derived and compared with the linear stability results of Howard & Gupta (1962). A discussion of the method and results is given in §9.

2. Equations of motion, Poisson bracket and constants of motion

We consider an inviscid, incompressible, axisymmetric flow in a cylindrical domain. By using cylindrical polar coordinates and writing the velocity $\boldsymbol{u} = (u_r, u_\theta, u_z)$ and pressure as p, the axisymmetric Euler equations are

$$\left.\begin{aligned} \frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} &= -\frac{\partial p}{\partial r}, \\ \frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z} &= 0, \\ \frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} &= -\frac{\partial p}{\partial z}, \end{aligned}\right\} \tag{2.1a}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0. \tag{2.1b}$$

Rather than working with the primitive variables $(\boldsymbol{u}, p)$ directly, it is convenient to define a Stokes stream function ψ to satisfy (2.1 b) identically, thus we set

$$(u_r, u_z) = \frac{1}{r}\left(-\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial r}\right).$$

Moreover, because the flow is axisymmetric, we can reduce the remaining three equations to two equations in swirl function $\kappa = ru_\theta$ and azimuthal vortex density

$$\chi = \frac{1}{r}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \equiv -\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right) \overset{\text{def}}{\equiv} \mathscr{L}\psi.$$

The evolution equations for κ and χ can then be written, from (2.1 a), as

$$\frac{\partial\kappa}{\partial t} + \{\psi, \kappa\} = -2\nu y \mathscr{L}\kappa, \tag{2.2a}$$

$$\frac{\partial\chi}{\partial t} + \{\psi, \chi\} = \frac{1}{4y^2}\frac{\partial}{\partial z}(\kappa^2) - 2\nu y\mathscr{L}\chi + 4\nu\frac{\partial\chi}{\partial y}, \tag{2.2b}$$

where $\{f, g\}$ is the canonical Poisson bracket or jacobian:

$$\{f, g\} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}, \tag{2.3}$$

and $y = \frac{1}{2}r^2$ is the new radial coordinate. In equations (2.2) we have included the viscous terms (multiplied by kinematic viscosity ν) for later reference. Because $\chi = \mathscr{L}\psi$, (2.2*a*, *b*) constitute a pair of coupled equations for the dependent variables κ and χ. Pressure has disappeared as a dynamical variable, essentially because of our taking the curl to obtain χ.

We shall study the nonlinear stability of equilibrium solutions of (2.2*a*, *b*) with viscosity set to zero, in the cylindrical domain $\mathrm{D} = \{(r, z) \mid 0 \leqslant r \leqslant R, 0 \leqslant z \leqslant L\}$. We assume that the equilibrium flows satisfy the no penetration boundary condition on $r = R$ and are L-periodic in the z-direction. This class includes z-dependent flows as well as columnar flows of the form $\boldsymbol{u} = (0, V_\mathrm{e}(r), W_\mathrm{e}(r))$ or $(\kappa, \chi) = (\kappa_\mathrm{e}(y), \chi_\mathrm{e}(y))$. The class of perturbations considered satisfies $\delta u_r(R, z) = 0$ and $\delta \boldsymbol{u}(r, 0) = \delta \boldsymbol{u}(r, L)$: they are thus, like the equilibrium flows, axially periodic and they satisfy the no-penetration boundary condition on the outer boundary $r = R$. Finally, we require that the perturbations preserve flow rate, thus admitting a vorticity formulation of the hamiltonian, below.

The most important conserved quantity for (2.2*a*, *b*) is the kinetic energy

$$H = \frac{1}{2}\int_\mathrm{D} (u_r^2 + u_\theta^2 + u_z^2)\,\mathrm{d}^3x = \frac{1}{2}\int_\mathrm{D}\left(\psi\chi + \frac{\kappa^2}{2y}\right)\mathrm{d}^3x + \frac{Q}{2}\int_0^L u_z|_R\,\mathrm{d}z, \tag{2.4}$$

where Q is the volume flow rate through the cylindrical domain. We remark that $\psi = 0$ on $r = 0$ and $\psi = Q/2\pi$ on $r = R$, and $\kappa = 0$ on $r = 0$ because of the singular coordinate system; however, κ is generally non-zero on $r = R$. In fact, equations (2.2) inherit a non-canonical hamiltonian structure from the general (non-axisymmetric) Euler equations, and (2.4) is the hamiltonian, as we now show. Lewis *et al.* (1986, equation (5.3)), derive a general Lie–Poisson bracket for incompressible, inviscid fluid flow, including boundary terms. For fixed boundaries, the general bracket is

$$\{F(\boldsymbol{u}), G(\boldsymbol{u})\} = \int_\mathrm{D} \boldsymbol{\omega}\cdot\left(\frac{\delta^\wedge F}{\delta \boldsymbol{u}} \times \frac{\delta^\wedge G}{\delta \boldsymbol{u}}\right)\mathrm{d}^3x + \int_\mathrm{D}\left(\boldsymbol{\omega}\cdot\left(\frac{\delta^\wedge F}{\delta \boldsymbol{u}} \times \frac{\delta^\vee G}{\delta \boldsymbol{u}}\right) + \nabla(f_* - f)\cdot\frac{\delta^\vee G}{\delta \boldsymbol{u}} - F \leftrightarrow G\right)\mathrm{d}A, \tag{2.5a}$$

where the functional derivatives are defined via

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F(\boldsymbol{u} + \epsilon\delta\boldsymbol{u}) \equiv \int_\mathrm{D} \frac{\delta^\wedge F}{\delta \boldsymbol{u}}\cdot\delta\boldsymbol{u}\,\mathrm{d}^3x + \int_{\partial\mathrm{D}} \frac{\delta^\vee F}{\delta \boldsymbol{u}}\cdot\delta\boldsymbol{u}\bigg|_{\partial\mathrm{D}}\mathrm{d}A, \tag{2.5b}$$

and $F \leftrightarrow G$ indicates that the preceding terms are repeated with F and G interchanged. Here $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{u}$ is the vorticity vector. The volume part of the functional derivative (2.5*b*), $\delta^\wedge F/\delta\boldsymbol{u}$, lies in the space of divergence-free vector fields parallel to the boundary ∂D. The projection of a general vector field on D onto the component of the Helmholtz (or Weyl–Hodge) decomposition parallel to the boundary is denoted P. The surface part, $\delta^\vee F/\delta\boldsymbol{u}$, is a vector field on ∂D whose component normal to ∂D is determined to within the gradient of an harmonic function. Finally, the term $\nabla(f_* - f)$ is an abbreviation for

$$(I-P)\left((\nabla \boldsymbol{u})\cdot\frac{\delta^{\vee} F}{\delta \boldsymbol{u}} - \left(\frac{\delta^{\vee} F}{\delta \boldsymbol{u}}\cdot\nabla\right)\boldsymbol{u}\right),$$

which lies in the space of gradient vector fields on D, orthogonal via the L_2 pairing to the space of divergence-free vector fields parallel to ∂D. We note that we must have either $\delta^{\vee}F/\delta\boldsymbol{u} = \mathbf{0}$ or $\delta^{\vee}G/\delta\boldsymbol{u} = \mathbf{0}$ to avoid difficulties associated with the bracket (2.5*a*). This restriction turns out not to matter to the arguments presented in this paper; see Lewis *et al.* (1986).

Integrating the bracket (2.5*a*) by parts, rewriting in terms of κ and χ and setting spatial derivatives with respect to the azimuthal coordinate equal to zero yields the axisymmetric bracket $\{\cdot,\cdot\}^{\odot}$, given below. We obtain

$$\{F,G\}^{\odot} = \int_D \left[\chi\left\{\frac{\delta G}{\delta\chi},\frac{\delta F}{\delta\chi}\right\} + \kappa\left\{\frac{\delta G}{\delta\chi},\frac{\delta F}{\delta\kappa}\right\} + \kappa\left\{\frac{\delta G}{\delta\kappa},\frac{\delta F}{\delta\chi}\right\}\right]\mathrm{d}^3x$$
$$+\int_0^L\left[\kappa\left(\frac{\delta F}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\delta G}{\delta\kappa}\right) - \frac{\delta G}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\delta F}{\delta\kappa}\right)\right)\right]_{r=0}^{r=R} 2\pi\,\mathrm{d}z$$
$$+\int_0^L\frac{\partial\kappa}{\partial z}\left[\frac{\delta G}{\delta v_b}\frac{\partial}{\partial z}\left(\frac{\delta F}{\delta\kappa}\right) - r\frac{\delta G}{\delta w_b}\frac{\delta F}{\delta\kappa}\right]\Bigg|_{r=0}^{r=R} 2\pi\,\mathrm{d}z, \tag{2.6}$$

where $\{f,g\}$ denotes the canonical bracket (2.3) and the functional derivatives are defined via

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F(\boldsymbol{u}+\epsilon\delta\boldsymbol{u}) = \int_D\left(\frac{\delta F}{\delta\chi}\delta\chi + \frac{\delta F}{\delta\kappa}\delta\kappa\right)\mathrm{d}^3x + \int_{\partial D}\frac{\delta F}{\delta \boldsymbol{u}_b}\cdot\delta\boldsymbol{u}\,\mathrm{d}A, \tag{2.7}$$

where

$$\frac{\delta F}{\delta\boldsymbol{u}_b} = \left(\frac{\delta F}{\delta u_b},\frac{\delta F}{\delta v_b},\frac{\delta F}{\delta w_b}\right)$$

is an element of the space of divergence-free vector fields parallel to ∂D. In interpreting (2.7) we must keep in mind the relationship between the primitive variable $\boldsymbol{u}$, and the derived variables (χ,κ) and, although it does not appear explicitly, ψ. In particular, the boundary velocity field $\boldsymbol{u}_b$ is derivable from (χ,κ) and indeed the boundary term can be rewritten in terms of those variables. In (2.6) we have assumed that $\delta F/\delta\boldsymbol{u}_b = 0$, consonant with the restrictions on the application of the parent bracket (2.5*a*). We remark that we require $\delta F/\delta\chi \sim$ constant or $\delta F/\delta\chi \sim r, r\to 0$, for one to be able to apply the bracket (2.6) to the function F, otherwise one has problems with singularities on the axis of the coordinate system. Thus, for example, $\delta H/\delta\chi = \psi$, $\delta H/\delta\kappa = \kappa/2y$ and $\delta H/\delta\boldsymbol{u}_b = (0,0,Q/2\pi R)$. Note that we have used preservation of flowrate in this calculation. The axisymmetric functions form a subalgebra within the larger Lie–Poisson structure. In fact the reduced structure implicit in $\{\cdot,\cdot\}^{\odot}$ is that of a semi-direct product (Holm *et al.* 1985), a structure that arises in a finite dimensional context in studies of rigid bodies with gravity (cf. Holmes & Marsden 1983).

Equipped with the axisymmetric bracket, the equations of motion (2.2*a, b*), with viscosity set to zero, are derived from the hamiltonian (2.4) just as in the canonical case. For a general function F we write

$$\dot{F} = \{F,H\}^{\odot}$$

or

$$\int_{\mathrm{D}}\left(\frac{\delta F}{\delta\chi}\dot{\chi}+\frac{\delta F}{\delta\kappa}\dot{\kappa}\right)\mathrm{d}^3x=\int_{\mathrm{D}}\left[\chi\left\{\frac{\delta H}{\delta\chi},\frac{\delta F}{\delta\chi}\right\}+\kappa\left\{\frac{\delta H}{\delta\chi},\frac{\delta F}{\delta\kappa}\right\}+\kappa\left\{\frac{\delta H}{\delta\kappa},\frac{\delta F}{\delta\chi}\right\}\right]\mathrm{d}^3x$$

$$+\int_0^L\left[\kappa\left(\frac{\delta F}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\delta H}{\delta\kappa}\right)-\frac{\delta H}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\delta F}{\delta\kappa}\right)\right)\right]_{r=R}2\pi\,\mathrm{d}z,$$

$$+\int_0^L\frac{\partial\kappa}{\partial z}\left[\frac{\delta H}{\delta v_{\mathrm{b}}}\frac{\partial}{\partial y}\left(\frac{\delta F}{\delta\chi}\right)-r\frac{\delta H}{\delta w_{\mathrm{b}}}\frac{\delta F}{\delta\kappa}\right]\Bigg|_{r=R}2\pi\,\mathrm{d}z,$$

$$=\int_{\mathrm{D}}\left[\chi\left\{\psi,\frac{\delta F}{\delta\chi}\right\}+\kappa\left\{\psi,\frac{\delta F}{\delta\kappa}\right\}+\kappa\left\{\frac{\kappa}{2y},\frac{\delta F}{\delta\chi}\right\}\right]\mathrm{d}^3x,$$

$$+\int_0^L\left[\kappa\left(\frac{\delta F}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\kappa}{2y}\right)-\kappa\psi\frac{\partial}{\partial z}\left(\frac{\delta F}{\delta\kappa}\right)\right)\right]_{r=R}2\pi\,\mathrm{d}z-Q\int_0^L\frac{\delta F}{\delta\chi}\frac{\partial\kappa}{\partial z}\Bigg|_R\mathrm{d}z,$$

where a dot denotes differentiation with respect to time. Integrating by parts several times and using the boundary conditions yields

$$\dot{F}=\int_{\mathrm{D}}\left(\frac{\delta F}{\delta\chi}\left[-\{\psi,\chi\}+\left\{\kappa,\frac{\kappa}{2y}\right\}\right]+\frac{\delta F}{\delta\kappa}[-\{\psi,\kappa\}]\right)\mathrm{d}^3x.$$

Equations (2.2*a*, *b*), less the viscous terms, then emerge on equating the factors multiplying $\delta F/\delta\kappa$ and $\delta F/\delta\chi$ respectively.

There is a finely judged analogue between the hamiltonian formulation developed above and that for stratified, two-dimensional, Boussinesq flow given by Abarbanel *et al.* (1986). Those authors (§2) write the equations of motion in terms of (scalar) vorticity ω, density ρ and stream function ψ as

$$\frac{\partial\rho}{\partial t}+\{\psi,\rho\}=0, \tag{2.8a}$$

$$\frac{\partial\omega}{\partial t}+\{\psi,\omega\}=\left\{\frac{gz}{\rho^*},\rho\right\}\equiv\frac{g}{\rho_*}\frac{\partial\rho}{\partial x}, \tag{2.8b}$$

where ρ_* is a reference density and the problem is defined in the (x,z)-plane (z vertical) with the canonical bracket

$$\{f,g\}=\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}. \tag{2.9}$$

The analogy between (2.2–2.3) and (2.8–2.9) is clear. In the present problem, κ and χ play the roles of ρ and ω, y and z replace z and x and the additional density term $\{gz/\rho_*,\rho\}$ of (2.8*b*) is replaced by the swirl term $(1/4y^2)(\partial/\partial z)(\kappa^2)=\{\kappa,\kappa/2y\}$. Thus the centrifugal force due to swirl plays the role of the gravitational force in the stratified flow problem. Moreover, the bracket derived by Abarbanel *et al.* (1986, §7) is similar except for boundary terms to equation (2.6). Analogous boundary terms would appear in the stratified flow bracket if a more general class of disturbances were considered (cf. Holm 1986). We will return to this analogy in discussing our results.

In addition to energy, the axisymmetric nature of the problem and translation invariance in the z direction lead to the existence of additional constants of motion, some of which are, in fact, Casimirs (Holm *et al.* 1985). Such functions, C, satisfy

$$\{C,F\}^{\odot}=0 \tag{2.10}$$

when bracketed with any function F of κ and χ in the class considered. Because $\dot{C} = \{C, H\}^{\odot} = 0$, C is a constant of motion.

In this case, the constants of motion are:

$$C_{\mathrm{I}} = \int_{\mathrm{D}} y\chi \,\mathrm{d}^3x\text{: impulse;} \tag{2.11a}$$

$$C_{\mathrm{S}} = \int_{\mathrm{D}} j(\kappa) \,\mathrm{d}^3x\text{: generalized swirl;} \tag{2.11b}$$

and

$$C_{\mathrm{H}} = \int_{\mathrm{D}} \chi f(\kappa) \,\mathrm{d}^3x\text{: generalized helicity.} \tag{2.11c}$$

Here j and f are arbitrary real-valued functions that will be chosen later. It will be convenient to separate circulation $C_{\mathrm{c}} = \int_{\mathrm{D}} \chi \,\mathrm{d}^3x$ from generalized helicity as a special constant of motion. Moreover, integrating C_{c} by parts we find that C_{c} is the difference of circulations on the axis $r = 0$ and on the outer boundary $r = R$. Direct differentiation shows that

$$\Gamma_R = 2\pi \int_0^L w|_R \,\mathrm{d}z \quad \text{and} \quad \Gamma_0 = 2\pi \int_0^L w|_0 \,\mathrm{d}z$$

are each conserved. In application of the energy-Casimir method we exploit conservation of Γ_R by using it to balance the flow rate term of the hamiltonian when forcing the Arnol'd function to have a fixed point at an equilibrium configuration.

The reader can check $\mathrm{d}C/\mathrm{d}t \equiv 0$ for each of (2.11*a*–*c*); the Appendix gives a sample calculation for $C = C_{\mathrm{S}}$ to illustrate how the boundary terms appear in integrations by parts and why the precise class of functions is important. This calculation shows that generalized swirl, C_{S}, is in fact a Casimir with respect to functions F of the type mentioned above. The others of (2.11) are Casimirs only for more restricted classes of functions.

In the case of non-rotating flows, $\kappa \equiv 0$ and the Poisson bracket reduces to

$$\{F, G\}_{\chi} = \int_{\mathrm{D}} \chi \left\{\frac{\delta G}{\delta \chi}, \frac{\delta F}{\delta \chi}\right\} \mathrm{d}^3x, \tag{2.12}$$

the hamiltonian becomes $H = \int_{\mathrm{D}} \psi\chi \,\mathrm{d}^3x + \frac{1}{2}Q \int_0^L w|_R \,\mathrm{d}z$, and there is only a single equation of motion

$$\frac{\partial \chi}{\partial t} + \{\psi, \chi\} = 0. \tag{2.13}$$

A new constant of motion, the generalized vortex density

$$C_{\mathrm{v}} = \int_{\mathrm{D}} \Phi(\chi) \,\mathrm{d}^3x, \tag{2.14}$$

exists in this case. Here the analogy is to two-dimensional shear flow; see Holm (1986) and Arnol'd (1965, 1966, 1969).

Finally, in this section we discuss the equilibrium solutions of the inviscid evolution equations (2.2*a*, *b*):

$$\{\psi_{\mathrm{e}}, \kappa_{\mathrm{e}}\} = 0, \tag{2.15a}$$

$$\{\psi_{\mathrm{e}}, \chi_{\mathrm{e}}\} = \frac{1}{4y^2} \frac{\partial}{\partial z} (\kappa_{\mathrm{e}}^2). \tag{2.15b}$$

(Here and henceforth the subscript e denotes a quantity evaluated at equilibrium.) Equations (2.15) imply that two functional relations exist among the functions ψ_e, κ_e and χ_e. From (2.15*a*), because the (canonical) bracket vanishes identically, we can write

$$\psi_e = K(\kappa_e), \tag{2.16}$$

where $K: \mathbb{R} \rightarrow \mathbb{R}$ is a function of κ_e alone. Rewriting (2.15*b*) using the functional dependence (2.16), we have

$$\frac{1}{4y^2}\frac{\partial}{\partial z}(\kappa_e^2) = \frac{1}{4y^2}\frac{d(\kappa_e^2)}{d\psi_e}\frac{\partial \psi_e}{\partial z} = -\frac{d(\kappa_e^2)}{d\psi_e}\left\{\frac{1}{4y}, \psi_e\right\}$$
$$= \left\{\psi_e, \frac{1}{4y}\frac{d(\kappa_e^2)}{d\psi_e}\right\},$$

and thus (2.15*b*) becomes

$$\left\{\psi_e, \chi_e - \frac{1}{4y}\frac{d(\kappa_e^2)}{d\psi_e}\right\} = 0 \tag{2.17}$$

and we obtain a second functional dependence

$$\chi_e - \frac{1}{4y}\frac{d(\kappa_e^2)}{d\psi_e} = V(\psi_e), \tag{2.18}$$

where $V: \mathbb{R} \rightarrow \mathbb{R}$ is a function of ψ_e only. This is the analogue of Long's equation in planar Boussinesq flow (cf. Abarbanel *et al.* 1986, equation (2.13)). In specific examples, because χ_e, κ_e and ψ_e are given, K and V can be computed.

For a columnar flow, the second functional dependence (2.17) simplifies, because $\kappa_e = \kappa_e(y)$ is independent of z and we have

$$\chi_e = X(\psi_e); \tag{2.19}$$

the same relation holds in the non-rotating case $\kappa_e \equiv 0$, but (2.16) is void in that case.

3. Criteria for formal and nonlinear stability

Taking the hamiltonian and the constants of motion from the preceding section we form the Arnol'd function,

$$A(\chi, \kappa) = H(\chi, \kappa) + C(\chi, \kappa)$$
$$= \frac{1}{2}\int_D \left(\psi\chi + \frac{\kappa^2}{2y}\right) d^3x + \frac{Q}{2}\int_0^L w|_R \, dz + 2\pi c\int_0^L w|_R \, dz + \int_D j(\kappa)\, d^3x + \int_D \chi f(\kappa)\, d^3x. \tag{3.1}$$

Note that the boundary circulation $\Gamma_R = 2\pi \int_0^L w|_R \, dz$ has been separated from generalized helicity C_H and that impulse (2.11*a*) has not been included (it turns out to make no contribution in what follows: we remark that Abarbanel *et al.* (1986) ignore the analogous constant of motion $\int_D z\omega \, dx\, dz$ in their study of two-dimensional Boussinesq flow). The constant c and the functions j and f are as yet undetermined.

In this section we shall obtain sufficient conditions for the convexity of A in the neighbourhood of a general (non-columnar) equilibrium flow $(\chi_e(y, z), \kappa_e(y, z))$. We first choose c, j and f so that the first variation $\delta A(\chi, \kappa)$ vanishes at the equilibrium point (χ_e, κ_e). We then examine the second variation $\delta^2 A(\chi_e, \kappa_e)$: if $\delta^2 A$ is (positive- or negative-) definite then, as described in Holm *et al.* (1985), we have formal stability. This implies stability of the linearized hamiltonian evolution equations. To obtain nonlinear stability we must check that the Arnol'd functional is convex near (χ_e, κ_e): that it is bounded above and below in terms of a quadratic 'energy-like' norm. Although formal stability implies nonlinear stability in finite-dimensional systems, there exist counter-examples in infinite dimensions (Ball & Marsden 1984) and definiteness of $\delta^2 A(\chi_e, \kappa_e)$ is necessary, but not sufficient, for nonlinear stability. Here the analysis is additionally complicated by an essential indefiniteness in $\delta^2 A$, which we shall overcome by applying a high-wavenumber cut off in the next section.

From (3.1), the first variation is

$$\delta A(\chi,\kappa) = DA(\chi,\kappa)\cdot(\delta\chi,\delta\kappa) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} A(\chi+\epsilon\,\delta\chi, \kappa+\epsilon\,\delta\kappa)$$

$$= \int_D \left[\frac{1}{2}\left(\psi\,\delta\chi+\chi\,\delta\psi+\frac{\kappa}{y}\delta\kappa\right)+j'(\kappa)\,\delta\kappa\right.$$

$$\left.+f(\kappa)\,\delta\chi+\chi f'(\kappa)\,\delta\kappa\right]d^3x+\left(\frac{Q}{2}+2\pi c\right)\int_0^L \delta w\,|_R\,dz. \tag{3.2}$$

Integrating the second term of (3.2) by parts, we may regroup terms to obtain

$$\delta A(\chi,\kappa) = \int_D\left[(\psi+f(\kappa))\,\delta\chi+\left(\frac{\kappa}{2y}+j'(\kappa)+\chi f'(\kappa)\right)\delta\kappa\right]d^3x+(Q+2\pi c)\int_0^L \delta w\,|_R\,dz. \tag{3.3}$$

Here a prime denotes differentiation with respect to argument: $j'(\kappa) = dj/d\kappa$, etc. Inserting the equilibrium solution, we obtain conditions on c, j and f:

$$\psi_e+f(\kappa_e) = 0, \tag{3.4a}$$

$$(\kappa_e/2y)+j'(\kappa_e)+\chi_e f'(\kappa_e) = 0, \tag{3.4b}$$

$$Q+2\pi c = 0, \tag{3.4c}$$

which must be satisfied if δA is to vanish at equilibrium.

We now use the functional relations (2.16) and (2.18). Equations (2.16) and (3.4*a*) allow us to write

$$f(\lambda) = -K(\lambda), \tag{3.5}$$

where K will be a known function of its argument in specific examples, and thus f can be determined. Hence we may write $f'(\kappa_e)$ and (3.4*b*) becomes

$$(\kappa_e/2y)+j'(\kappa_e)-\chi_e K'(\kappa_e) = 0. \tag{3.6}$$

From (2.18) we have

$$\kappa_e/2y = [\chi_e - V(\psi_e)]\,(dK/d\kappa)\,(\kappa_e), \tag{3.7}$$

which may be substituted into (3.6) to yield

$$j'(\kappa_e) = V(\psi_e)\,K'(\kappa_e),$$

or, upon integration

$$j(\kappa_e) = \int_0^{\kappa_e} V(\lambda)\,K'(\lambda)\,d\lambda. \tag{3.8}$$

Selecting f, c and j according to (3.4c), (3.5) and (3.8), we have the first variation

$$\delta A = \int_D \left[(\psi - K(\kappa))\, \delta\chi + \left(\frac{\kappa}{2y} + V(K(\kappa))\, K'(\kappa) - \chi K'(\kappa) \right) \delta\kappa \right] d^3x, \tag{3.9}$$

which vanishes at equilibrium, as required.

Next we compute the second variation from (3.9):

$$\delta^2 A = D^2 A(\chi, \kappa) \cdot (\delta\chi, \delta\kappa)^2 = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \delta A(\chi + \epsilon\,\delta\chi, \kappa + \epsilon\,\delta\kappa)$$

$$= \int_D \left(\delta\psi\, \delta\chi - K'(\kappa)\, \delta\chi\, \delta\kappa + \frac{1}{2y} \delta\kappa\, \delta\kappa + V'(K(\kappa))\, [K'(\kappa)]^2\, \delta\kappa\, \delta\kappa \right.$$
$$\left. + V(K(\kappa))\, K''(\kappa)\, \delta\kappa\, \delta\kappa - K'(\kappa)\, \delta\chi\, \delta\kappa - \chi K''(\kappa)\, \delta\kappa\, \delta\kappa \right) d^3x. \tag{3.10}$$

Rewriting the first term with the operator $\mathscr{L}$ defined in §2:

$$\int_D \delta\psi\, \delta\chi\, d^3x = \int_D \delta\chi\, \mathscr{L}^{-1}\, \delta\chi\, d^3x, \tag{3.11}$$

we have, at equilibrium

$$\delta^2 A(\chi_e, \kappa_e) = \int_D (\delta\chi, \delta\kappa)$$
$$\times \begin{bmatrix} \mathscr{L}^{-1} & -K'(\kappa_e) \\ -K'(\kappa_e) & \left[\frac{1}{2y} + V'(K(\kappa_e))\, (K'(\kappa_e))^2 + (V(K(\kappa_e)) - \chi_e)\, K''(\kappa_e) \right] \end{bmatrix} \begin{pmatrix} \delta\chi \\ \delta\kappa \end{pmatrix} d^3x. \tag{3.12}$$

The lower-right matrix entry may be simplified, by using the functional relations (2.16)–(2.18) to give

$$\begin{bmatrix} \mathscr{L}^{-1} & -K'(\kappa_e) \\ -K'(\kappa_e) & \dfrac{\partial\chi_e}{\partial\kappa_e} \dfrac{d\psi_e}{d\kappa_e} + \dfrac{\kappa_e}{2y^2} \dfrac{\partial y}{\partial\kappa_e} \end{bmatrix}. \tag{3.13}$$

In this calculation we use the implicit functional relation $y = y(\kappa_e, \chi_e)$; we also implicitly assume that this functional relation is monotone, so that the requisite inverse is uniquely defined. See §8 for examples.

Completing the square, from (3.13) we have

$$\delta^2 A_e = \int_D \left[\left(\surd(\mathscr{L}^{-1})\, \delta\chi - \frac{K'(\kappa_e)}{\surd\mathscr{L}^{-1}} \delta\kappa \right)^2 + \left(\frac{\partial\chi_e}{\partial\kappa_e} \frac{d\psi_e}{d\kappa_e} + \frac{\kappa_e}{2y^2} \frac{\partial y}{\partial\kappa_e} - \frac{1}{\mathscr{L}^{-1}} (K'(\kappa_e))^2 \right) (\delta\kappa)^2 \right] d^3x. \tag{3.14}$$

The first term makes it clear that $\delta^2 A_e$ cannot be negative definite and a sufficient condition for positive definiteness is

$$\frac{\partial\chi_e}{\partial\kappa_e} \frac{d\psi_e}{d\kappa_e} + \frac{\kappa_e}{2y^2} \frac{\partial y}{\partial\kappa_e} - \frac{1}{\mathscr{L}^{-1}} \left(\frac{d\psi_e}{d\kappa_e} \right)^2 \geqslant 0. \tag{3.15}$$

This is our condition for formal stability. The interpretation of the operators $\surd\mathscr{L}^{-1}$ and $\mathscr{L}^{-1}$ in (3.14)–(3.15) will become clear in §4 and in the examples below. Essentially, we will need to bound the expression $\int_D \delta\chi\, \mathscr{L}^{-1}\, \delta\chi\, d^3x$ above and below.

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A problem is immediately apparent. Inequality (3.15) cannot be expected to hold in general for the simple reason that the eigenvalues of the operator

$$\mathscr{L} = -\frac{1}{r^2}\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)$$

have no upper bound. Consequently the eigenvalues of $\mathscr{L}^{-1}$ accumulate on zero and $1/\mathscr{L}^{-1}$ is unbounded. Thus, perturbations $\delta\chi$ having high spatial wavenumbers will make the third term of (3.15) large and negative. This suggests that we limit the class of perturbations, or modify the model appropriately, to provide a high-wavenumber cut off. We address this issue in §4, after obtaining criteria for nonlinear stability.

We require convexity estimates of the form

$$Q(\Delta\chi,\Delta\kappa) \leqslant A(\chi_e+\Delta\chi,\kappa_e+\Delta\kappa)-A(\chi_e,\kappa_e)-\mathrm{D}A(\chi_e,\kappa_e)\cdot(\Delta\chi,\Delta\kappa)$$

$$\stackrel{\text{def}}{=} \hat{A}(\Delta\chi,\Delta\kappa) = \hat{H}(\Delta\chi,\Delta\kappa)+\hat{C}(\Delta\chi,\Delta\kappa), \quad (3.16)$$

where $\Delta\chi$, $\Delta\kappa$ are finite (but small) perturbations and Q is a quadratic functional that satisfies $Q(v)>0$ for $v\neq 0$. If such a functional Q can be found, then we may define the norm $\|(x,y)\| = \surd[Q(x,y)]$. If, moreover, A is continuous in this norm at (χ_e,κ_e), then it follows that the equilibrium is nonlinearly stable. A sufficient condition for continuity is that $\hat{A}$ is bounded above: $\hat{A}(x,y)\leqslant C\|(x,y)\|^2$ (see Holm *et al.* 1985, §2).

As a first step to bounding $\hat{A}(\Delta\chi,\Delta\kappa)$ from below, we rewrite $\hat{A}$ with the aid of Taylor's theorem. Expansion of the integrand of $A(\chi,\kappa)$ about an equilibrium flow configuration results in the quadratic form with 'remainder':

$$2\hat{A}(\Delta\chi,\Delta\kappa) = \int_{\mathrm{D}} (\Delta\chi,\Delta\kappa)\begin{bmatrix} \mathscr{L}^{-1} & \dfrac{\partial^2 a}{\partial\chi\,\partial\kappa}(\chi_e,\kappa_e) \\ \dfrac{\partial^2 a}{\partial\chi\,\partial\kappa}(\chi_e,\kappa_e) & \dfrac{\partial^2 a}{\partial\kappa^2}(\chi_e,\kappa_e)\end{bmatrix}\begin{pmatrix}\Delta\chi\\ \Delta\kappa\end{pmatrix}\mathrm{d}^3x$$

$$+\mathscr{O}(|\Delta\chi|^3,|\Delta\chi|^2|\Delta\kappa|,|\Delta\chi|\,|\Delta\kappa|^2,|\Delta\kappa|^3), \quad (3.17)$$

where $a(\chi,\kappa)$ is the integrand of the Arnol'd functional. Restriction to small but finite perturbations will ultimately allow us to neglect the cubic and higher-order terms in (3.17).

Completing the square, we have

$$2\hat{A}(\Delta\chi,\Delta\kappa) = \int_{\mathrm{D}}\left[\left\{\surd(\mathscr{L}^{-1})\,\Delta\chi+\frac{(\partial^2 a/\partial\chi\,\partial\kappa)\,(\chi_e,\kappa_e)}{\surd\mathscr{L}^{-1}}\Delta\kappa\right\}^2\right.$$

$$\left.+\left\{\frac{\partial^2 a}{\partial\kappa^2}(\chi_e,\kappa_e)-\frac{[(\partial^2 a/\partial\chi\,\partial\kappa)\,(\chi_e,\kappa_e)]^2}{\mathscr{L}^{-1}}\right\}(\Delta\kappa)^2\right]\mathrm{d}^3x. \quad (3.18)$$

Now, suppose that one is able to find bounds t^-, t^+ and m^-, m^+ such that

$$t^- \leqslant \left|\frac{\partial^2 a}{\partial\chi\,\partial\kappa}(\chi_e,\kappa_e)\right| \leqslant t^+, \quad (3.19)$$

and

$$0<m^-\leqslant\frac{\partial^2 a}{\partial\kappa^2}(\chi_e,\kappa_e)\leqslant m^+<\infty, \quad (3.20)$$

then a lower bound to (3.18) is

$$Q(\Delta\chi,\Delta\kappa) \equiv \int_D \left[\left\{\sqrt{\mathscr{L}^{-1}}\,\Delta\chi + \frac{1}{\sqrt{\mathscr{L}^{-1}}}\frac{\partial^2 a}{\partial\chi\,\partial\kappa}(\chi_e,\kappa_e)\,\Delta\kappa\right\}^2 + \left\{m^- - \frac{(t^+)^2}{\mathscr{L}^{-1}}\right\}(\Delta\kappa)^2\right] d^3x. \tag{3.21}$$

The quadratic form (3.21) is a suitable norm provided

$$m^- > (t^+)^2/\mathscr{L}^{-1}. \tag{3.22}$$

As for formal stability, a high-wavenumber cut off will be required to meet this condition.

The existence of the upper bound m^+ ensures continuity of the norm $\|\Delta\chi,\Delta\kappa\|^2 \equiv Q(\Delta\chi,\Delta\kappa)$. Thus there is a limit on the growth of perturbations that depends only on their initial size:

$$\|\Delta\chi(t),\Delta\kappa(t)\|^2 \leqslant \frac{m^+}{m^-}\|\Delta\chi(t=0),\Delta\kappa(t=0)\|^2. \tag{3.23}$$

Computation of the integrand matrix entries of (3.17) for the case of general axisymmetric flow yields the following. Provided there exist bounds t^-, t^+ and m^-, m^+ such that

$$t^- \leqslant |-K'(\kappa_e)| \leqslant t^+, \tag{3.24}$$

$$0 < m^- \leqslant \left(\frac{\partial\chi_e}{\partial\kappa_e}\frac{d\psi_e}{d\kappa_e} + \frac{\kappa_e}{2y^2}\frac{\partial y}{\partial\kappa_e}\right) \leqslant m^+ < \infty, \tag{3.25}$$

with

$$m^- > (t^+)^2/\mathscr{L}^{-1}, \tag{3.26}$$

then the flow (χ_e,κ_e) is nonlinearly stable to axisymmetric disturbances in the norm (3.21).

4. The high-wavenumber cut off: stability theorems

In this section we restrict our attention to perturbations of vortex density $\Delta\chi$ whose expansions in eigenfunctions of $\mathscr{L}$ terminate at finite order. This permits us to obtain sharp, albeit conditional, criteria for formal and nonlinear stability and to replace the inequalities (3.15) and (3.24) by specific statements. To justify such a procedure we appeal to the physical origin of our problem. To profit from the hamiltonian formulation we have chosen to study stability of inviscid flows. In a real fluid, the arbitrarily large velocity gradients that accompany high-spatial-wavenumber vortex density (and swirl) perturbations necessitate inclusion of viscous effects, no matter how small the kinematic viscosity may be. Thus, inviscid stability results can only be expected to yield physically reasonable predictions for perturbations of bounded wavenumber, such as those defined below. Naturally, it is more satisfying mathematically if conditions exist under which arbitrary perturbations are stable, but, as we have seen, this is impossible in the present case. Abarbanel *et al.* (1986) also found it impossible in the analogous problem for stratified flow and they devised alternative model equations that are 'blind' to high wavenumbers by modifying the laplacian operator in that problem.

In this paper we prefer to cut off the wavenumber of disturbances in χ, with the following justification. One can perform a linear stability analysis of equations (2.2*a*, *b*) about a columnar equilibrium. It is possible to show that high-wavenumber disturbances in χ decay exponentially with a time constant of order wavenumber squared multiplied by viscosity. Thus we conclude that dangerous disturbances are typically of low to moderate wavenumber, even in weakly viscous flows. Thus, although our analysis is inviscid, it seems reasonable to neglect such high-wavenumber disturbances, which would be rapidly damped in physical situations.

Here the trouble revolves around finding a lower bound for the quantity

$$\int_D \Delta\chi \mathscr{L}^{-1} \Delta\chi \, d^3x. \tag{4.1}$$

The eigenvalues μ_i of $\mathscr{L}^{-1}$ are the inverses of the eigenvalues λ_j of

$$\mathscr{L} = -\frac{1}{r^2}\left(\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right).$$

Separation of variables leads to Whittaker's equation in the radial coordinate: Sturm–Liouville theory implies that the eigenvalues λ_i are a countable sequence, unbounded above:

$$0 < \lambda^- \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \tag{4.2a}$$

and thus,

$$0 = \mu_0 \leqslant \mu_1 \leqslant \mu_2 \ldots \leqslant 1/\lambda^-, \tag{4.2b}$$

where the smallest eigenvalue is given by $\lambda^- = 4\pi^2/R^4$ (recall that R is the radius of the domain D). The theory also gives a simple estimate for eigenvalues $\lambda_{n,m}$ and eigenfunctions of large orders n, m. If n is the number of internal zeros in $r \in (0, R)$ and m the number of internal zeros in $z \in (0, L)$, then we have

$$\lambda_N = \max_{0 \leqslant m < M} \lambda_{n,m} \approx \frac{4\pi^2}{R^4}(N+2)^2 \quad \text{for} \quad n \gg M \gg 1, \tag{4.3}$$

where $N = Mn$.

Now if we admit only those variations in vortex density which can be expressed in terms of the eigenfunctions ϕ_j of $\mathscr{L}$ as the finite sum

$$\Delta\chi(r,z) = \sum_{j=1}^{N} c_j \phi_j(r,z), \tag{4.4}$$

we may write

$$\int_D \Delta\chi \mathscr{L}^{-1} \Delta\chi \, d^3x = \int_D \left(\sum_{i,j}^{N} c_i \phi_i \mathscr{L}^{-1} c_j \phi_j\right) d^3x$$

$$= \int_D \left(\sum_{i,j}^{N} \frac{c_i c_j}{\lambda_j} \phi_i \phi_j\right) d^3x = \sum_{i,j}^{N} \frac{c_i c_j}{\lambda_j} \int_D \phi_i \phi_j \, dx.$$

We therefore obtain the bounds

$$\frac{1}{\lambda_N}\int_D (\Delta\chi)^2 d^3x \leqslant \int_D \Delta\chi \mathscr{L}^{-1} \Delta\chi \, d^3x \leqslant \frac{1}{\lambda^-}\int_D (\Delta\chi)^2 d^3x, \tag{4.5}$$

the lower one depending upon the order N of our wavenumber cut off. Thus, restricting ourselves to finite wavenumber perturbations of the form (4.4), we may replace $\mathscr{L}^{-1}$ by its lower bound $1/\lambda_N$ or upper bound $1/\lambda^-$, as appropriate, in the inequalities (3.15) and (3.26) to obtain the following.

THEOREM 4.1. (*Formal stability of axisymmetric flow*). *If an equilibrium flow* $(\chi_e, \kappa_e, \psi_e)$ *is such that*

$$\frac{\partial \chi_e}{\partial \kappa_e}\frac{d\psi_e}{d\kappa_e}+\frac{\kappa_e}{2y^2}\frac{\partial y}{\partial \kappa_e}-\lambda_N\left(\frac{d\psi_e}{d\kappa_e}\right)^2 > 0 \tag{4.6}$$

at every point $(y, z) \in [0, \frac{1}{2}R^2] \times [0, L]$ *of* D *for an eigenvalue* λ_N *of* $\mathscr{L}$, *N large, then* $(\chi_e, \kappa_e, \psi_e)$ *is formally stable to all axisymmetric perturbations in* κ *and to axisymmetric perturbations in* χ *expressible as* (4.4).

THEOREM 4.2. (*Nonlinear stability of axisymmetric flow*). *If an equilibrium flow* (χ_e, κ_e) *is such that constants* t^-, t^+ *and* m^-, m^+ *exist satisfying* (3.24) *and* (3.25) *with*

$$m^- > \lambda_N (t^+)^2, \tag{4.7}$$

where λ_N *is an eigenvalue of* $\mathscr{L}$, *N large, then* (χ_e, κ_e) *is nonlinearly stable in the norm* (3.21) *to axisymmetric disturbances in* κ *and* χ, *provided that the* χ *disturbance can be expanded as in* (4.4).

In special situations it is possible to obtain unconditional stability results without resorting to a wavenumber cut off (see §6, below).

5. COLUMNAR FLOWS

In this and the next section we address some special cases of equilibrium flows. Columnar flows, whose velocity fields are independent of z, form an important special class of equilibrium swirling flows. It is easy to check that any velocity field of the form $\boldsymbol{u} = (0, u_\theta(r), u_z(r))$ satisfies equation (2.1), regardless of the functions u_θ, u_z. Analytical solutions of this form are therefore plentiful, whereas to our knowledge analytical solutions involving swirling flows with non-trivial z-dependence are not numerous (asymptotic solutions do exist; cf. Leibovich & Randall (1973 *a*, *b*) and Leibovich (1978, 1984)). The examples we discuss in §8 are therefore drawn from the class of columnar flows.

As we pointed out in §2, the functional relations in the columnar case are

$$\chi_e = X(\psi_e), \quad \psi_e = K(\kappa_e), \tag{5.1a}$$

derived from the equilibrium conditions $\{\psi_e, \kappa_e\} = \{\psi_e, \chi_e\} = 0$, because the expression $\partial/\partial z(\kappa_e^2) \equiv 0$ if κ_e is independent of z. (This also implies that the radial coordinate y is functionally dependent on κ_e (or χ_e, or ψ_e); we therefore write

$$y = Y(\kappa_e). \tag{5.1b}$$

We seek conditions sufficient for $\delta A_e = 0$ (3.4 *a*, *b*) after restoring the impulse $c_2 \int_D y\chi \, d^3x$ to the Arnol'd functional to obtain

$$\psi_e + c_2 y + f(\kappa_e) = 0, \tag{5.2a}$$

$$(\kappa_e/2y) + j'(\kappa_e) + \chi_e f'(\kappa_e) = 0, \tag{5.2b}$$

and (3.4 *c*). Using the columnar functional relations (5.1 *a*, *b*) we obtain

$$f(\kappa_e) = -c_2 Y(\kappa_e) - K(\kappa_e), \tag{5.3a}$$

$$j'(\kappa_e) = -[\kappa_e/2Y(\kappa_e)] + X(\psi_e)[c_2 Y'(\kappa_e) + K'(\kappa_e)], \tag{5.3b}$$

so that, making these substitutions for f, j', the first variation is

$$\delta A = \int_{\mathrm{D}} \left[\{\psi + c_2(y - Y(\kappa)) - K(\kappa)\}\,\delta\chi + \left\{ \frac{K}{2}\left(\frac{1}{y} - \frac{1}{Y(\kappa)}\right) + (X(K(\kappa)) - \chi)\,(c_2\, Y'(\kappa) + K'(\kappa)) \right\} \delta\kappa \right] \mathrm{d}^3x. \quad (5.4)$$

As required, $\delta A(\chi_{\mathrm{e}}, \kappa_{\mathrm{e}}) = 0$.

The second variation is

$$\delta^2 A = \int_{\mathrm{D}} \left[\delta\psi\,\delta\chi - 2(c_2\, Y'(\kappa) + K'(\kappa))\,\delta\psi\,\delta\kappa + \left\{ \frac{1}{2}\left(\frac{1}{y} - \frac{1}{Y(\kappa)} + \frac{\kappa Y'(\kappa)}{Y^2(\kappa)}\right) + X'(K(\kappa))\,K'(\kappa)\,(c_2\, Y'(\kappa) + K'(\kappa)) + (X(K(\kappa)) - \chi)\,c_2\, Y''(\kappa) + K''(\kappa)) \right\} \delta\kappa\,\delta\kappa \right] \mathrm{d}^3x; \quad (5.5)$$

which, upon evaluation at equilibrium, simplifies to

$$\delta^2 A_{\mathrm{e}} = \int_{\mathrm{D}} \left[\delta\chi\,\mathscr{L}^{-1}\,\delta\chi - 2(c_2\, Y'(\kappa_{\mathrm{e}}) + K'(\kappa_{\mathrm{e}}))\,\delta\chi\,\delta\kappa + \left\{ \frac{\kappa_{\mathrm{e}}\, Y'(\kappa_{\mathrm{e}})}{2y^2} + X'(K(\kappa_{\mathrm{e}}))\,K'(\kappa_{\mathrm{e}})\,(c_2\, Y'(\kappa_{\mathrm{e}}) + K'(\kappa_{\mathrm{e}})) \right\} \delta\kappa\,\delta\kappa \right] \mathrm{d}^3x, \quad (5.6)$$

because $\chi_{\mathrm{e}} = X(\psi_{\mathrm{e}}) = X(K(\kappa_{\mathrm{e}}))$ and $y = Y(\kappa_{\mathrm{e}})$ at equilibrium. Completing the square, as in §3, (5.6) becomes

$$\delta^2 A_{\mathrm{e}} = \int_{\mathrm{D}} \left[\left(\sqrt{\mathscr{L}^{-1}}\,\delta\chi - \frac{e_{12}}{\sqrt{\mathscr{L}^{-1}}}\,\delta\kappa \right)^2 + \left(e_{22} - \frac{e_{12}^2}{\mathscr{L}^{-1}} \right) (\delta\kappa)^2 \right] \mathrm{d}^3x, \quad (5.7\,a)$$

where

$$e_{12} = c_2\, Y'(\kappa_{\mathrm{e}}) + K'(\kappa_{\mathrm{e}}), \quad (5.7\,b)$$

and

$$e_{22} = \frac{\kappa_{\mathrm{e}}}{2y^2}\, Y'(\kappa_{\mathrm{e}}) + X'(K(\kappa_{\mathrm{e}}))\,K'(\kappa_{\mathrm{e}})\,e_{12}.$$

It is evident that a sufficient condition for formal stability is

$$e_{22} - (e_{12}^2/\mathscr{L}^{-1}) > 0. \quad (5.8)$$

To satisfy (5.8) we will have to use the high-wavenumber cut off of §4, but before addressing this problem we rewrite (5.8) in a fashion reminiscent of the Howard & Gupta (1962) criterion for linear stability. Because formal stability implies linear stability we expect a result comparable to theirs. Certainly for positive definiteness of $\delta^2 A_{\mathrm{e}}$ we require $e_{22} > 0$. If we choose the free parameter $c_2 = 0$, so that $e_{12} = K'(\kappa_{\mathrm{e}})$, this condition may be rewritten

$$\frac{\kappa_{\mathrm{e}}}{2y^2}\frac{\mathrm{d}y}{\mathrm{d}\kappa_{\mathrm{e}}} + \frac{\mathrm{d}\chi_{\mathrm{e}}}{\mathrm{d}\psi_{\mathrm{e}}}\left(\frac{\mathrm{d}\psi_{\mathrm{e}}}{\mathrm{d}\kappa_{\mathrm{e}}}\right)^2 > 0.$$

Upon multiplication by $(\mathrm{d}\kappa_{\mathrm{e}}/\mathrm{d}y)^2$ this yields

$$\frac{\kappa_{\mathrm{e}}}{2y^2}\frac{\mathrm{d}\kappa_{\mathrm{e}}}{\mathrm{d}y} + \frac{\mathrm{d}\chi_{\mathrm{e}}}{\mathrm{d}y}\frac{\mathrm{d}\psi_{\mathrm{e}}}{\mathrm{d}y} > 0. \quad (5.9)$$

Because for a columnar flow $u_z = W_e = d\psi_e/dy$ and $\chi_e = -dW_e/dy$, (5.9) may be rewritten again:

$$\frac{\kappa_e}{2y^2}\frac{d\kappa_e}{dy} - W_e\frac{d^2W_e}{dy^2} > 0. \tag{5.10}$$

In the same variables the Richardson number criterion for linear stability of Howard & Gupta (1962) is

$$J \equiv \frac{\kappa_e}{2y^2}\frac{d\kappa_e}{dy}\cdot\frac{1}{(dW_e/dy)^2} > \frac{1}{4}$$

or

$$\frac{\kappa_e}{2y^2}\frac{d\kappa_e}{dy} - \frac{1}{4}\left(\frac{dW_e}{dy}\right)^2 > 0. \tag{5.11}$$

Note the similarity between (5.10) and (5.11). In specific examples the similarity is even closer; see §8. It is significant that the factor $\frac{1}{4}$ occurs in the linear stability criterion compared with 1 in the formal stability criterion: precisely the same ratios emerge in the stratified flow analysis of Abarbanel *et al.* (1986, §2, equations (2.47–2.48)).

We now turn to the convexity estimate required for nonlinear stability. As in §3 (equations (3.17–3.23)), we obtain a quadratic functional form for $2\hat{A}(\Delta\chi, \Delta\kappa)$. The required bounds t^-, t^+ and m^-, m^+ are

$$t^- \leqslant |-c_2\, Y'(\kappa_e) - K'(\kappa_e)| \leqslant t^+, \tag{5.12}$$

$$\begin{aligned} 0 < m^- &\leqslant [(\kappa_e/2y^2)\, Y'(\kappa_e) + X'(\psi_e)\, K'(\kappa_e)\,(c_2\, Y'(\kappa_e) + K'(\kappa_e))] \\ &\leqslant m^+ < \infty, \end{aligned} \tag{5.13}$$

over the range of κ_e, ψ_e. The lower bound for $2\hat{A}(\Delta\chi, \Delta\kappa)$ is then

$$Q(\Delta\chi, \Delta\kappa) = \int_D \left[\left(\surd(\mathcal{L}^{-1})\,\Delta\chi + \frac{(c_2\, Y'(\kappa_e) + K'(\kappa_e))}{\surd\mathcal{L}^{-1}}\Delta\kappa\right)^2 + \left(m^- - \frac{(t^+)^2}{\mathcal{L}^{-1}}\right)(\Delta\kappa)^2\right] d^3x. \tag{5.14}$$

The quadratic form $Q(\Delta\chi, \Delta\kappa)$ is a suitable norm for the disturbance $(\Delta\chi, \Delta\kappa)$ provided that it is positive definite. Application of the high-wavenumber cut off yields the condition

$$m^- > \lambda_N (t^+)^2 \tag{5.15}$$

for $Q(\Delta\chi, \Delta\kappa)$ given by (5.14) to be positive definite. The upper bound m^+ of (5.13) ensures continuity of the Arnol'd functional in the norm (5.14) and thus nonlinear stability. Arguments analogous to those in §4 yield the following.

THEOREM 5.1. (*Formal stability of columnar flow*). *If a columnar equilibrium flow* $(\chi_e, \kappa_e, \psi_e)$ *is such that*

$$e_{22} - \lambda_N e_{12}^2 > 0 \tag{5.16}$$

at every point in D, *with* e_{12}, e_{22} *defined in* (5.7 *b*) *and* λ_N *an eigenvalue of* $\mathcal{L}$, *N large, then* $(\chi_e, \kappa_e, \psi_e)$ *is formally stable to all axisymmetric perturbations in* κ *and to axisymmetric perturbations in* χ *expressible as* (4.4).

THEOREM 5.2. (*Nonlinear stability of columnar flow*). *If a columnar equilibrium flow* (χ_e, κ_e) *is such that there exist bounds* t^-, t^+, *and* m^-, m^+ *satisfying* (5.12) *and* (5.13) *with* $m^- > \lambda_N(t^+)^2$, *for* λ_N *an eigenvalue of the operator* $\mathcal{L}$, *N large, then* (χ_e, κ_e) *is nonlinearly stable in the norm* (5.14) *to axisymmetric disturbances in* κ *and* χ, *provided that the latter can be expanded as in* (4.4).

We note that, although the equilibrium flows must be columnar, the disturbances may have nontrivial z-dependence.

6. Non-rotating flows

A second special class of flows are those in which there is non-trivial r- and z-dependence, but the flow has no swirl, i.e. $u_\theta = 0$ and $\kappa = 0$. The disturbances we consider here will also have no swirl, so that the variable κ vanishes from the problem. Although such flows are of less practical interest than swirling flows, the structure of the constants of motion is significantly different and unconditional stability results can be obtained: one does not require a high-wavenumber cut off. The analysis in this section closely parallels Arnol'd's (1965, 1966, 1969) studies of planar fluid flows, because we are concerned only with the vortex density χ_e and the stream function ψ_e (χ_e plays the role of the scalar vorticity ω_e in Arnol'd's study).

Referring to §2, we have the single equation of motion (2.13) and the associated Arnol'd functional consisting of kinetic energy plus generalized vortex density (equation (2.14)):

$$A(\chi) = \frac{1}{2}\int_D \psi\chi\,d^3x + \frac{Q}{2}\int_0^L w|_R\,dz + \int_D \Phi(\chi)\,d^3x + 2\pi c\int_0^L w|_R\,dz. \tag{6.1}$$

As before, to choose Φ we require that the first variation,

$$\begin{aligned}\delta A(\chi) &= \int_D [\tfrac{1}{2}\psi(\delta\chi + \chi\,\delta\psi) + \Phi'(\chi)\,\delta\chi]\,d^3x + \left(\frac{Q}{2} + 2\pi c\right)\int_0^L \delta w|_R\,dz \\ &= \int_D [\psi + \Phi'(\chi)]\,\delta\chi\,d^3x + (Q + 2\pi c)\int_0^L \delta w|_R\,dz\end{aligned} \tag{6.2}$$

vanish at equilibrium χ_e. This implies that

$$\Phi'(\chi_e) = -\psi_e, \quad c = -Q/2\pi, \tag{6.3a, b}$$

and because $\{\psi_e, \chi_e\} = 0$ at equilibrium (equation (2.13)) we have the functional relation $\psi_e = \Psi(\chi_e)$ and (6.3a) becomes

$$\Phi(\chi_e) = -\int_0^{\chi_e} \Psi(\lambda)\,d\lambda. \tag{6.4}$$

From (6.2) we find the second variation

$$\delta^2 A(\chi) = \int_D [\delta\chi\,\mathscr{L}^{-1}\,\delta\chi + \Phi''(\chi)\,(\delta\chi)^2]\,d^3x, \tag{6.5}$$

and thus for formal stability we require that the expression

$$\int_D [\delta\chi\,\mathscr{L}^{-1}\,\delta\chi + \Phi''(\chi_e)\,(\delta\chi)^2]\,d^3x \tag{6.6}$$

be definite. Here we have a choice; although the first term is certainly non-negative, the second term can take either sign and thus we can seek either positive or negative definiteness. This leads to the conditions

$$\Phi''(\chi_e) > 0 \text{ (positive definiteness)}, \tag{6.7a}$$

$$-\Phi''(\chi_e) > 1/\lambda^- = R^4/4\pi^2 \text{ (negative definiteness)}. \tag{6.7b}$$

In the latter we have employed the smallest of the eigenvalues of $\mathscr{L}$ from §4, but note that λ^- does not depend on wavenumber. (It does of course depend on the radius R of the domain.)

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Nonlinear stability follows, as above, from a consideration of finite perturbations $\Delta\chi$. We examine the functional

$$\hat{A}(\Delta\chi) = A(\chi_e+\Delta\chi) - A(\chi_e) - DA(\chi_e)\cdot\Delta\chi$$
$$= \int_D [\Delta\chi\,\mathscr{L}^{-1}\Delta\chi + \Phi''(\chi_e)(\Delta\chi)^2]\,d^3x + \mathscr{O}(|\Delta\chi|^3). \tag{6.8}$$

We must consider the positive and negative definite cases separately. In the former, it will suffice to find constants

$$0 < c^- \leqslant \Phi''(\chi_e) \leqslant c^+ < \infty \tag{6.9}$$

for then we can choose the norm

$$\|\Delta\chi\| = \left(\int_D [\Delta\chi\,\mathscr{L}^{-1}\Delta\chi + c^-(\Delta\chi)^2]\,d^3x\right)^{\frac{1}{2}} \tag{6.10}$$

and it will follow that

$$\|\Delta\chi\|^2 \leqslant \hat{A}(\Delta\chi) \leqslant (c^+/c^-)\,\|\Delta\chi\|^2. \tag{6.11}$$

This will imply nonlinear stability.

In the negative definite case we will require constants $c^+ > c^- > 0$ such that

$$-\infty < -c^+ \leqslant \Phi''(\chi_e) \leqslant -c^- < 0 \quad \text{and} \quad c^- > 1/\lambda^-. \tag{6.12}$$

In that case (6.11) will hold with the norm

$$\|\Delta\chi\| = \left(\int_D [c^-(\Delta\chi)^2 - \Delta\chi\,\mathscr{L}^{-1}\Delta\chi]\,d^3x\right)^{\frac{1}{2}}. \tag{6.13}$$

Summarizing, we have the following.

THEOREM 6.1. *If an axisymmetric, non-rotating equilibrium flow (χ_e, ψ_e) satisfies either*

$$\text{(i)} \quad 0 < c^- \leqslant \Phi''(\chi_e) \leqslant c^+ < \infty, \tag{6.14a}$$

or

$$\text{(ii)} \quad -\infty < -c^+ \leqslant \Phi''(\chi_e) \leqslant -c^- < 0 \quad \text{and} \quad c^- > 1/\lambda^- = R^4/4\pi^2 \tag{6.14b}$$

for positive constants $c^+ > c^- > 0$ at each point on the domain D, then (χ_e, ψ_e) is nonlinearly stable to all axisymmetric non-rotating disturbances.

7. Annular domains

The formal and nonlinear stability analyses developed for cylindrical domains adapt easily to annular domains of the form

$$D' = \{(r,z)\,|\,0 < R_1 \leqslant r \leqslant R_2;\, 0 \leqslant z \leqslant L\}. \tag{7.1}$$

The boundary condition $u_r = 0$ at $r = 0$ is replaced by $u_r = 0$ at $r = R_1$ and the only other significant change is in the bounds obtained for the eigenvalues of $\mathscr{L}$. Note that the solutions of $\psi = \mathscr{L}^{-1}\chi$ are unique despite the annular domain; such a domain is simply connected in the (y,z)-plane, although not in three dimensions. The minimum eigenvalue λ^- becomes

$$\lambda^- = 4\pi^2/(R_2^2 - R_1^2)^2, \tag{7.2}$$

and the asymptotic form analogous to (4.3) for high-order eigenvalues becomes

$$\lambda'_N \approx \frac{4\pi^2(N+2)^2}{(R_2^2-R_1^2)^2}. \tag{7.3}$$

Equations of motion and conserved quantities all remain unchanged, apart from the integration domain D′, and analogues of Theorems 4.1–4.2, 5.1–5.2 and 6.1 exist, with λ_N replaced by λ'_N of (7.2) and the eigenfunctions of (4.4) replaced by eigenfunctions ϕ'_j of $\mathscr{L}$ defined on D′.

8. Examples

We now illustrate the stability theory developed above by applying it to several examples. We know of few examples of analytic solutions to the axisymmetric Euler equations in finite, cylindrical domains having both non-zero swirl and non-trivial, periodic z-dependence; these we shall consider together with numerically generated solutions in another paper. See also Wan (1988) for analyses of the Hill's spherical vortex and toroidal vortices, and Benjamin (1976) for remarks about the stability of vortex rings: these are both axisymmetric solutions with z-dependence, but in infinite domains and without swirl. Here our examples are all of columnar flows.

We shall give three examples. First we consider an exponential axial velocity profile with zero swirl. Application of the results of §6 permits subdivision of the parameter space into regions in which one can and cannot prove nonlinear stability of the kind in Theorem (6.1). We follow with an example of a columnar swirling flow: the same exponential jet or wake with solid-body rotation. The critical stability parameter for this example is a swirl ratio; we demonstrate that increasing the rotation of the baseflow has a stabilizing effect. This effect has already been noted in linear stability studies (cf. Howard & Gupta 1962).

Finally, we consider the more important example of the exponential wake or jet with a Burgers vortex profile for the swirl velocity. This flow matches experimental data for flows which are prone to vortex breakdown, as described by Garg & Leibovich (1979), for example. One may view the axisymmetric form of (small-amplitude) vortex breakdown as a finite disturbance on a columnar flow. The stability of such a baseflow to such a disturbance is governed by the analysis of §5. Moreover, the special form of this last example allows us to obtain nonlinear stability criteria independent of any high-wavenumber cut off.

Example 1. *Non-rotating exponential jet or wake.* This example has the velocity field $(u_r, u_\theta, u_z) = (0, 0, W_e)$ with the profile

$$W_e = 1+\delta\, e^{-\alpha y}, \tag{8.1}$$

where $\alpha > 0, \delta \neq 0$ are parameters. Equation (8.1) describes a two-parameter family of equilibria of the axisymmetric Euler equations, with stream function and vortex density

$$\psi_e = y + (\delta/\alpha)(1-e^{-\alpha y}), \tag{8.2}$$

$$\chi_e = \alpha\delta\, e^{-\alpha y}, \tag{8.3}$$

and $\kappa_e = 0$. One may solve (8.3) for y in terms of χ_e and substitute into (8.2) to obtain

$$\psi_e = \Psi(\chi_e) = -\frac{1}{\alpha}\ln\left(\frac{\chi_e}{\alpha\delta}\right) + \frac{\delta}{\alpha}\left(1-\frac{\chi_e}{\alpha\delta}\right). \tag{8.4}$$

The function $\Phi''(\chi_e)$ of Theorem 6.1 is therefore

$$\Phi''(\chi_e) = -\frac{d\Psi}{d\chi_e}(\chi_e) = \frac{e^{\alpha y}+\delta}{\alpha^2\delta}. \tag{8.5}$$

Now, if $\delta > 0$, we have

$$0 < \frac{1+\delta}{\alpha^2\delta} \leqslant \Phi''(\chi_e) \leqslant \frac{e^{\frac{1}{2}\alpha R^2}+\delta}{\alpha^2\delta} < \infty. \tag{8.6}$$

These provide the bounds c^-, c^+ in (6.14*a*); thus we have nonlinear stability for all $\delta > 0$. This establishes stability of jet-like profiles.

For $-1 < \delta < 0$, we have

$$-\infty < \frac{-(e^{\frac{1}{2}\alpha R^2}-|\delta|)}{\alpha^2|\delta|} \leqslant \Phi''(\chi_e) \leqslant -\frac{(1-|\delta|)}{\alpha^2|\delta|} < 0. \tag{8.7}$$

Thus we have the bounds $-c^+, -c^-$ of (6.14*b*) and thus for $\delta \in (-1, 0)$, the flow (8.1) is nonlinearly stable to disturbances without swirl provided

$$|\delta| < \left(1+\frac{\alpha^2 R^4}{4\pi^2}\right)^{-1}. \tag{8.8}$$

The last inequality comes from requiring $c^- > 1/\lambda^- = R^4/4\pi^2$. Thus wake-like profiles with sufficiently weak velocity decrement are nonlinearly stable. However, when $\delta \leqslant -1$ one can see from (8.6) and (8.7) that we cannot demonstrate definiteness. Note that for $\delta = 0$ the profile degenerates into a uniform flow $w_e \equiv 1$ and the function Ψ is not defined. Figure 1 illustrates the criterion (8.8) and shows examples of stable velocity profiles.

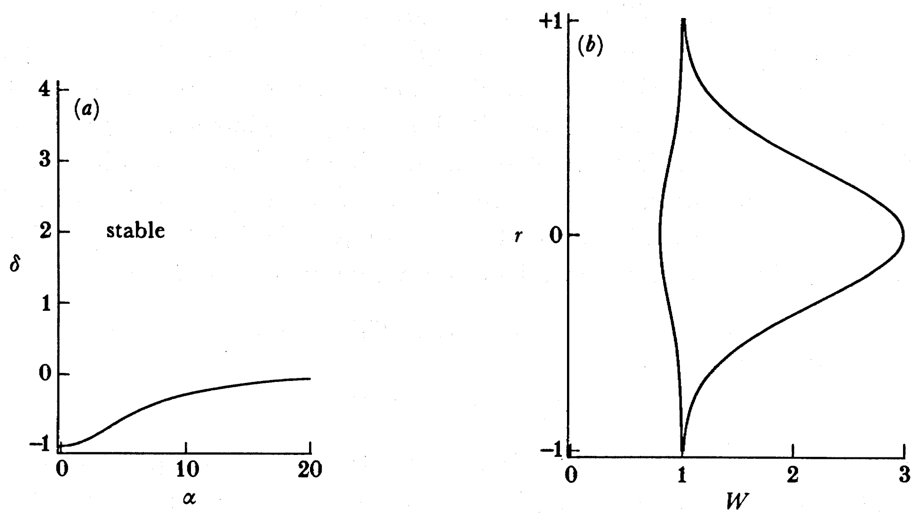

FIGURE 1. (*a*) Stability diagram for non-rotating exponential jet or wake, $R = 1$. (*b*) Stable axial velocity profiles of non-rotating exponential jet ($R = 1$, $\delta = 2$, $\alpha = 10$) and wake ($R = 1$, $\delta = -0.2$, $\alpha = 10$).

Example 2. *Rotating exponential jet or wake.* For our second example we add a solid-body rotation $u_\theta = \frac{1}{2}\beta r$ or

$$\kappa_e = \beta y \tag{8.9}$$

to the profile (8.1–8.3) of the previous example. The theory of §5 is now applicable. We start by considering the linear Richardson number criterion (5.11), due to Howard & Gupta (1962). Computing from (8.1–8.3) and (8.9) we have

$$J \equiv \frac{\kappa_e}{2y^2}\frac{d\kappa_e}{dy}\left(\frac{dW_e}{dy}\right)^{-2} = \frac{\beta^2}{2\alpha^2\delta^2}\cdot\frac{e^{2\alpha y}}{y} > \frac{1}{4} \tag{8.10}$$

for linear stability. The function $y\,e^{-2\alpha y}$ achieves its (unique) maximum at $y = 1/2\alpha$, thus if this point lies in the domain D we have

$$\Omega^2 > \alpha/4e, \quad R \geqslant \alpha^{-\frac{1}{2}}, \tag{8.11a}$$

or otherwise

$$\Omega^2 > \alpha^2R^2/(4e^{\alpha R^2}), \quad R < \alpha^{-\frac{1}{2}}, \tag{8.11b}$$

where $\Omega = |\beta/\delta|$ is the swirl ratio. Satisfaction of (8.11*a* or *b*) guarantees linear stability. We note that the radius $r = \alpha^{-\frac{1}{2}}$ is the point at which $d^2W_e/dr^2 = 0$: the inflection point.

We next consider formal stability, applying the criterion (5.16) $e_{22} - \lambda_N e_{12}^2 > 0$. Here

$$e_{12} = c_2\frac{dy}{d\kappa_e} + \frac{d\psi_e}{d\kappa_e} = \frac{c_2}{\beta} + \frac{1+\delta\,e^{-\alpha y}}{\beta} \tag{8.12a}$$

and

$$e_{22} = \frac{\kappa_e}{2y^2}\frac{dy}{d\kappa_e} + \frac{d\chi_e}{d\kappa_e}e_{12} = \frac{1}{2y} - \alpha^2\frac{\delta}{\beta}e^{-\alpha y}\left(\frac{c_2+1+\delta\,e^{-\alpha y}}{\beta}\right) \tag{8.12b}$$

from (5.7*b*) by using the functional relations derived from equations (8.1–8.3) and (8.9), for the flow at hand. Thus, to obtain the best criterion for any given wavenumber cut off λ_N, we must maximize the minimum value of

$$\frac{e_{22}}{e_{12}^2} = \frac{(\beta^2/2y) - \alpha^2\delta\,e^{-\alpha y}(1+c_2+\delta\,e^{-\alpha y})}{(1+c_2+\delta\,e^{-\alpha y})^2} \tag{8.13}$$

over the domain D ($y \in [0, \frac{1}{2}R^2]$) by the best choice of c_2, given α, β, δ. It is awkward to give general criteria; however, making the special choice of the free constant $c_2 = -1$, (8.13) becomes

$$\left(\frac{\beta}{\delta}\right)^2\frac{e^{2\alpha y}}{2y} - \alpha^2, \tag{8.14}$$

which takes its minimum value at $y = 1/2\alpha$. Thus from (5.16) sufficient, but not necessarily optimal, criteria for formal stability are

$$\left(\frac{\beta}{\delta}\right)^2 = \Omega^2 > \frac{\alpha}{e} + \frac{\lambda_N}{\alpha e}, \quad R \geqslant \alpha^{-\frac{1}{2}}, \tag{8.15a}$$

and

$$\Omega^2 > (\alpha^2 + \lambda_N)\,R^2\,e^{-\alpha R^2}, \quad R < \alpha^{-\frac{1}{2}}. \tag{8.15b}$$

These should be compared with the linear Howard–Gupta criteria of (8.11*a*, *b*). As in the analogous stratified flow results of Abarbanel *et al.* (1986, §4) the criteria differ by a factor of $\frac{1}{4}$ and addition of the 'eigenvalue' term to the formal criteria. Of course the latter are more stringent, because formal stability implies linear stability, but not vice versa (Holm *et al.* 1985).

Finally, to obtain sufficient conditions for nonlinear stability, we examine inequalities (5.12–5.13) and (5.15), obtaining

$$\left.\begin{aligned} t^- \leqslant \left|\frac{1+c_2+\delta\,\mathrm{e}^{-\alpha y}}{\beta}\right| \leqslant t^+ \\ 0 < m^- \leqslant \left[\frac{1}{2y} - \alpha^2\frac{\delta}{\beta}\,\mathrm{e}^{-\alpha y}\left(\frac{1+c_2+\delta\,\mathrm{e}^{-\alpha y}}{\beta}\right)\right] \leqslant m^+ < \infty \\ m^- > \lambda_N\, t^{+2}. \end{aligned}\right\} \tag{8.16}$$

In view of the term $1/2y$ in e_{22}, we cannot choose the upper bound $m^+ < \infty$. However, because the swirl function perturbations $\Delta\kappa$ belong to a class of functions for which the integral $\int_D (\Delta\kappa)^2/2y\,\mathrm{d}^3x < \infty$ is defined ($\Delta\kappa \sim y$ as $y \to 0$), the continuity arguments depending on the upper bound may be modified slightly with the same conclusions for $m^+ < \infty$. Thus our task is to choose c_2, depending on α, β, δ, such that the ratio m^-/t^{+2} is maximized. Once more it is difficult to give general results, but we can obtain a simple criterion by selecting $c_2 = -1$, as before. We then have

$$t^+ = |\delta/\beta| = \Omega^{-1} \tag{8.17a}$$

and can take

$$m^- = \min_{y\in[0,\frac{1}{2}R^2]}\left[\frac{1}{2y} - \frac{\alpha^2}{\Omega^2}\,\mathrm{e}^{-2\alpha y}\right] > 0$$

$$= \frac{1}{R^2} - \frac{\alpha^2}{\Omega^2}\,\mathrm{e}^{-\alpha R^2}, \quad \text{for} \quad R < \alpha^{-\frac{1}{2}}. \tag{8.17b}$$

Thus a sufficient condition for nonlinear stability of the exponential profile with solid-body rotation is

$$\Omega^2 > (\alpha^2\,\mathrm{e}^{-\alpha R^2} + \lambda_N)\,R^2; \quad R < \alpha^{-\frac{1}{2}}. \tag{8.18}$$

Note that (8.18) is more stringent than (8.15 *b*).

In special cases we can make optimum choices of c_2 and thus improve (8.15) and (8.18). We indicate this procedure for Example 3, below.

Example 3. *Burgers vortex with exponential jet or wake.* In this final example we consider the equilibrium solution given by

$$\left.\begin{aligned} W_\mathrm{e} = 1+\delta\,\mathrm{e}^{-\alpha y}, \quad \chi_\mathrm{e} = \alpha\delta\,\mathrm{e}^{-\alpha y} \\ \kappa_\mathrm{e} = \beta(1-\mathrm{e}^{-\alpha y}), \quad \psi_\mathrm{e} = y + \frac{\delta}{\alpha}(1-\mathrm{e}^{-\alpha y}), \end{aligned}\right\} \tag{8.19}$$

i.e. we take the axial profile of the previous examples and add an exponential swirl profile. In this case the Howard & Gupta (1962) Richardson number criterion (5.11) gives, after some calculation

$$(\beta/\delta) = \Omega^2 > \alpha y^2/2(\mathrm{e}^{\alpha y} - 1). \tag{8.20}$$

Maximizing the right-hand side we obtain *ca.* $1/3.08828\alpha$ at $y \approx 1.59362/\alpha$; thus sufficient conditions for linear stability are

$$\Omega^2 > 1/3.08828\alpha \quad \text{if} \quad R \geqslant (3.18724/\alpha)^{\frac{1}{2}}, \tag{8.21a}$$

and

$$\Omega^2 > \alpha R^4/8(\mathrm{e}^{\frac{1}{2}\alpha R^2} - 1) \quad \text{if} \quad R < (3.18724/\alpha)^{\frac{1}{2}}. \tag{8.21b}$$

Again the swirl ratio Ω is a key parameter.

As in the previous examples, for formal stability we use the functional relation (8.19) to write $y = -1/\alpha \ln[(1-\kappa_e)/\beta]$ and $\chi_e = \alpha\delta[1-(\kappa_e/\beta)]$. Thus we obtain

$$e_{12} = c_2\frac{dy}{d\kappa_e}+\frac{d\psi_e}{d\kappa_e} = c_2\frac{e^{\alpha y}}{\alpha\beta}+\frac{(\delta+e^{\alpha y})}{\alpha\beta}$$

$$= \frac{\delta}{\alpha\beta}\left(1+\frac{(1+c_2)\,e^{\alpha y}}{\delta}\right), \tag{8.22a}$$

$$e_{22} = \frac{\kappa_e}{2y^2}\frac{dy}{d\kappa_e}+\frac{d\chi_e}{d\kappa_e}e_{12} = \frac{\beta(1-e^{-\alpha y})}{2y^2}\frac{e^{\alpha y}}{\alpha\beta}-\frac{\alpha\delta}{\beta}e_{12}$$

$$= \left[\frac{e^{\alpha y}-1}{2\alpha y^2}-\left(\frac{\delta}{\beta}\right)^2\left(1+\frac{(1+c_2)\,e^{\alpha y}}{\delta}\right)\right]. \tag{8.22b}$$

Thus by Theorem 5.1 a sufficient condition for formal stability is

$$\Omega^2\left(\frac{e^{\alpha y}-1}{2\alpha y^2}\right)-\left(1+\frac{(1+c_2)\,e^{\alpha y}}{\delta}\right) > \frac{\lambda_N}{\alpha^2}\left(1+\frac{(1+c_2)\,e^{\alpha y}}{\delta}\right) \tag{8.23}$$

for all $y\in(0,\tfrac{1}{2}R^2)$, where λ_N is the cut-off eigenvalue. A simple sufficient condition is found, as above, by taking $c_2 = -1$, in which case (8.23) may be rewritten

$$\Omega^2 > \frac{2\alpha y^2}{e^{\alpha y}-1}\left(1+\frac{\lambda_N}{\alpha^2}\right). \tag{8.24}$$

Noting that the common factor on the right-hand side of (8.24) is just four times that of (8.20), we have

$$\Omega^2 > \frac{1}{0.77207\alpha}\left(1+\frac{\lambda_N}{\alpha^2}\right) \quad \text{if} \quad R \geqslant (3.18724/\alpha)^{\frac{1}{2}} \tag{8.25a}$$

and

$$\Omega^2 > \frac{\alpha R^4}{2(e^{\frac{1}{2}\alpha R^2}-1)}\left(1+\frac{\lambda_N}{\alpha^2}\right) \quad \text{if} \quad R < (3.18724/\alpha)^{\frac{1}{2}}. \tag{8.25b}$$

Again the factor of $\tfrac{1}{4}$ appears in comparison with the linear stability criterion.

Nonlinear stability requires that we bound e_{12} and e_{22}:

$$t^- \leqslant \left|\frac{-\delta}{\alpha\beta}\left(1+\frac{(1+c_2)\,e^{\alpha y}}{\delta}\right)\right| \leqslant t^+, \tag{8.26a}$$

$$0 < m^- \leqslant \left[\frac{e^{\alpha y}-1}{2\alpha y^2}-\left(\frac{\delta}{\beta}\right)^2\left(1+\frac{(1+c_2)\,e^{\alpha y}}{\delta}\right)\right] \leqslant m^+ < \infty. \tag{8.26b}$$

With the special choice $c_2 = -1$ we may take $t^+ = |-\delta/\alpha\beta| = 1/\alpha\Omega\ (= t^-)$ and

$$m^- = \min_{y\in[0,\frac{1}{2}R^2]}\left[\frac{e^{\alpha y}-1}{2\alpha y^2}-\frac{1}{\Omega^2}\right] = 0.77207\alpha-\Omega^{-2}$$

$(R \geqslant (3.18724/\alpha)^{\frac{1}{2}})$ or $2(e^{\frac{1}{2}\alpha R^2}-1)/\alpha R^4-\Omega^{-2}$ $(R < (3.18724/\alpha)^{\frac{1}{2}})$. Finally, via inequality $m^- > \lambda_N t^{+2}$ of Theorem 5.2, this yields a sufficient condition for nonlinear stability identical to that for formal stability (8.25*a, b*). As in Example 2, we must appeal to the fact that, although $m^+ = \infty$ because of the term $(e^{\alpha y}-1)/2\alpha y^2$ in e_{22}, $\int_D (e^{\alpha y}-1)/2\alpha y^2\,(\Delta\kappa)^2\,d^3x$ is finite.

In the above, as in Example 2, we have sought simple criteria rather than optimizing our sufficient conditions by varying the free constant c_2. In specific flow situations, with α, β, δ

given, one may improve the criteria. We indicate how this can be done in the case of jet-like profiles ($\delta > 0$). Recall that we wish to choose c_2 so that m^-/t^{+2} is maximized to obtain the best criterion.

Let

$$t(c_2, y) = \left|-\frac{1}{\alpha \Omega}\left(1+\frac{(1+c_2)}{\delta}\,\mathrm{e}^{\alpha y}\right)\right|, \tag{8.27a}$$

$$m(c_2, y) = \left[\frac{\mathrm{e}^{\alpha y}-1}{2\alpha y^2}-\frac{1}{\Omega^2}\left(1+\left(\frac{1+c_2}{\delta}\right)\mathrm{e}^{\alpha y}\right)\right], \tag{8.27b}$$

(recall $\Omega = |\beta/\delta|$), in which case

$$t^+(c_2) = \max_{y\in[0,\frac{1}{2}R^2]} t(c_2, y); \quad m^-(c_2) = \min_{y\in[0,\frac{1}{2}R^2]} m(c_2, y). \tag{8.28}$$

We first observe that, because $\delta > 0$ for jet-like profiles, $t^+(c_2) > t^+(-1)$ for $c_2 > -1$ or $c_2 \ll -1$. Moreover, $m^-(c_2) < m^-(-1)$ for $c_2 > -1$. These conclusions follow from the fact that $(1+[(1+c_2)/\delta]\,\mathrm{e}^{\alpha y}) > 1$ for all y and $c_2 > -1$. While $m^-(c_2)$ increases as $c_2 \rightarrow -\infty$, the ratio $m^-(c_2)/t^+(c_2)^2$ is dominated by $t^{+2} \sim c_2^2$ and we therefore conclude that to improve upon $c_2 = -1$, we must pick $c_2 < -1$ but $|c_2|$ not too large. The optimum depends in a complicated way on α, δ and R, the domain radius, but in general the best choices seem to lie in range around $-(1+\delta)$. (When $c_2 = -(1+\delta)$, $t(c_2, y) = 0$ at $y = 0$.) For example, taking $\alpha = 2$, $\delta = 2$, $\Omega = |\beta/\delta| = 10$ and $R = 1$ we compute $m^-(-1)/t^+(-1)^2 = 683.2$ whereas for $c_2 = -2.076$ we have $m^-(c_2)/t^+(c_2)^2 = 3{,}227$. With the bound $\lambda_N = 4\pi^2(N+2)^2/R^4$ from (4.3), this choice permits us to prove stability to all wavenumbers $N \leqslant 7$, whereas $c_2 = -1$ only gives stability for $N \leqslant 2$. Here the small radius is important: one generally finds that $c_2 = -1$ is close to optimal for large $\frac{1}{2}\alpha R^2$.

Our final example is special in that we can obtain a nonlinear stability criterion that is, for physical purposes, independent of the high-wavenumber cut off that appears as the eigenvalue λ_N in (8.25). To do this we take a distinguished limit $R \rightarrow \infty$, $N \rightarrow \infty$ with $L_\Delta \equiv R/\sqrt{N}$ fixed. L_Δ is chosen to agree with the approximate spacing, near $r = 0$, of zeros in the eigenfunctions of $\mathscr{L}$ for N large (cf. §4). Specifically, consider a vortex density disturbance of the form $\Delta\chi = c\phi_N$, where ϕ_N is a high-order eigenfunction of $\mathscr{L}$ and c is a constant of $\mathcal{O}(\chi_e)$, with dimensions $(\text{length})^{-1}(\text{time})^{-1}$. The corresponding disturbance in axial velocity is $\Delta w \sim (R^2/N)\,\Delta\chi$. Thus an appropriate Reynolds number for the disturbance is

$$\mathrm{Re} \equiv \frac{L_\Delta \Delta w}{\nu} \sim \frac{R}{\sqrt{N}}\frac{R^2}{N}\frac{c}{\nu}, \tag{8.29}$$

where ν is the kinematic viscosity. This Reynolds number is of order one when $\nu \sim (R^3/N^{\frac{3}{2}})\,c$, or $\nu \sim L_\Delta^3 c$. Thus for $L_\Delta^3 < \nu/c$, the effects of viscosity predominate in the evolution of the disturbance velocity field. If we take the limit $R \rightarrow \infty$, $N \rightarrow \infty$ with $L_\Delta \equiv R/\sqrt{N} = \nu_*^{\frac{1}{3}}$, where $\nu_* = \nu/c$ is the kinematic viscosity divided by a representative vortex density of the equilibrium flow, we obtain a stability result for Example 3 that is independent of high-frequency cut off:

$$\Omega^2 > \frac{1}{0.77207\alpha}+\frac{4\pi^2}{0.77207\alpha^3\nu_*^{\frac{4}{3}}}. \tag{8.30}$$

This is a sufficient 'physical' condition for nonlinear stability of the Burgers vortex plus exponential jet or wake to all axisymmetric disturbances, provided that the amplitude of the disturbances decays as $r \rightarrow \infty$ sufficiently quickly for the norm for nonlinear stability to be

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finite. Certainly Example 3 is not the only case for which one may obtain stability results independent of high-frequency cut off.

Figure 2*a* shows the criterion (8.30) and figure 2*b* illustrates a swirl velocity profile that is stable when combined with the jet of figure 1*b*.

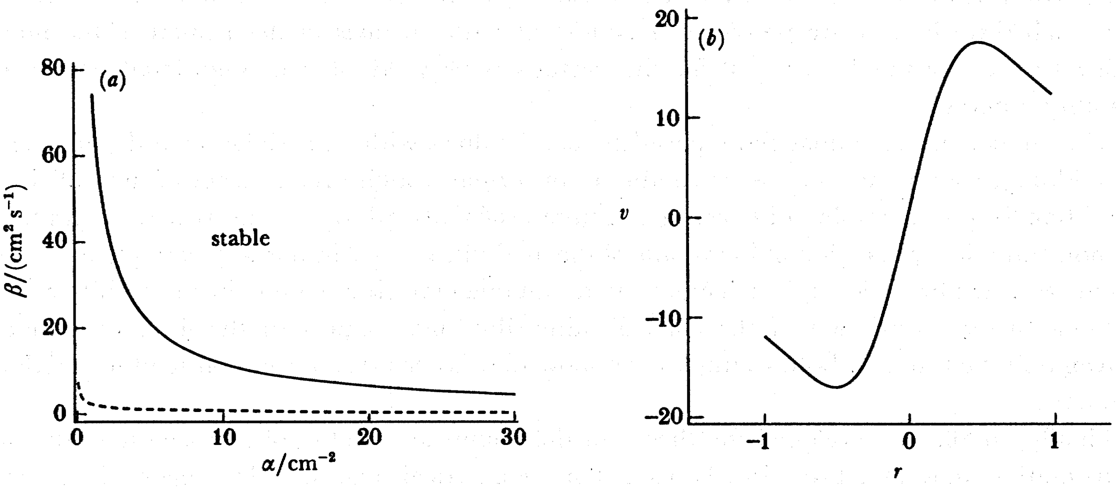

FIGURE 2. (*a*) Stability diagram for Burgers vortex with exponential jet in air (20 °C, $\delta = 2$ cm s^{-1}). Depicted are sufficient physical criteria for nonlinear stability and formal stability (broken line). (*b*) Stable-swirl velocity profile. This Burgers vortex ($\beta = 12$ cm^2 s^{-1}, $\alpha = 10$ cm^{-2}) is stable in combination with the exponential jet of figure 1*b*.

We remark that we have taken a distinguished limit in Example 3 in contrast to the simple discussion of viscous effects in §4. Here R and N tend to infinity with L_Δ fixed, whereas in §4 we let N tend to infinity to show that high-wavenumber disturbances decay.

9. Conclusions

We have considered the nonlinear stability of axisymmetric flows of an incompressible, inviscid fluid to axisymmetric disturbances via the energy-Casimir method of Arnol'd. Note that although nonlinear stability and formal stability results are obtained, which are stronger than linear stability, the stability is not global (Joseph 1976): we are restricted to finite but possibly arbitrarily small disturbances. Central to the method is the proof of the convexity of a conserved function of the dynamical variables at an equilibrium configuration of the system. For each of the three classes of axisymmetric flows we have considered, general, columnar and non-rotating, the conserved function was constructed from the total kinetic energy of the fluid and from constants of the motion corresponding to symmetries inherent in the given geometry.

These quantities are intimately related to the non-canonical hamiltonian structure as expressed in the Lie–Poisson bracket (2.6). The complexity of the boundary terms in (2.6) reflects our desire to accommodate as broad and as physically reasonable a class of perturbations as possible. In some earlier nonlinear stability analyses via the Arnol'd energy-Casimir method, strong assumptions were made on the classes of admissible perturbations. Such assumptions yield mathematically elegant formulations but are unduly restrictive and fail to capture the full range of physical behaviour. Examples include assumptions like preservation of boundary circulation by disturbances of plane parallel flows in Holm *et al.* (1984, 1985), and zero density variation on connected components of the boundary in the analysis of two-dimensional Boussinesq flow in Abarbanel *et al.* (1986).

For general axisymmetric flow, the convexity analysis of the appropriate Arnol'd function yields conditions on the baseflow sufficient to ensure nonlinear stability. Convexity generally fails, however, for disturbances in vortex density of arbitrarily large wavenumber. We argue that such disturbances are physically inconsistent with an inviscid description of the flow. Appeal to a wavenumber cut off for the perturbations yields sharp, conditional nonlinear stability criteria.

The analysis for columnar flows parallels that for flows with non-trivial axial dependence. In addition, we obtain a necessary condition for formal stability reminiscent of the sufficient condition for linear stability of Howard & Gupta (1962). In §8, we present two examples that demonstrate the application of the results of the analysis. For the important example of a flow prone to vortex breakdown, it is demonstrated that one may obtain a nonlinear stability result without further restriction of the class of admissible disturbances in the form of a high-wavenumber cut off. In both examples, we show that faster rates of rotation tend to stabilize the flow.

Finally, in §6, we developed the theory of the nonlinear stability of non-rotating flows to non-rotating axisymmetric disturbances. For geometrical reasons, this theory is closely analogous to Arnol'd's original anslysis of the nonlinear stability of planar, incompressible flow. Stability criteria for non-rotating axisymmetric flows are obtained without the need of a high-wavenumber cut off. An example illustrating the application of the result is also given in §8. The example demonstrates that exponential jets are stable, as are exponential wakes of 'shallow' profile.

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Appendix. Poisson bracket computations

In this Appendix, we give a sample calculation that proves that the generalized swirl (2.11 *b*) is a Casimir. First, we give a useful identity for the canonical *y*–*z* bracket (2.3), which may be proven by integration by parts

$$\int_D f\{g,h\}\,\mathrm{d}^3x = \int_D \{f,g\}\,h\,\mathrm{d}^3x + \int_0^{\frac{1}{2}R^2} fh\frac{\partial g}{\partial y}\bigg|_0^L 2\pi\,\mathrm{d}y - \int_0^L fh\frac{\partial g}{\partial z}\bigg|_0^{\frac{1}{2}R^2} 2\pi\,\mathrm{d}z. \qquad \text{(A 1)}$$

To show that C_S is a Casimir, we must show that $\{C_S, G\}^{\odot} = 0$ for any function $G(\chi,\kappa)$. We have

$$\begin{aligned}\{C_S, G\}^{\odot} = &\int_D \left[\chi\left\{\frac{\delta G}{\delta\chi},\frac{\delta C_S}{\delta\chi}\right\} + \kappa\left\{\frac{\delta G}{\delta\kappa},\frac{\delta C_S}{\delta\chi}\right\} + \kappa\left\{\frac{\delta G}{\delta\chi},\frac{\delta C_S}{\delta\kappa}\right\}\right]\mathrm{d}^3x \\ &+ \int_0^L \left[\kappa\left(\frac{\delta C_S}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\delta G}{\delta\kappa}\right) - \frac{\delta G}{\delta\chi}\frac{\partial}{\partial z}\left(\frac{\delta C_S}{\delta\kappa}\right)\right)\right]_{r=R} 2\pi\,\mathrm{d}z \\ &+ \int_0^L \frac{\partial\kappa}{\partial z}\left[\frac{\delta G}{\delta v_b}\frac{\partial}{\partial z}\left(\frac{\delta C_S}{\delta\chi}\right) - r\frac{\delta G}{\delta w_b}\frac{\delta C_S}{\delta\kappa}\right]\bigg|_{r=R} 2\pi\,\mathrm{d}z. \end{aligned} \qquad \text{(A 2)}$$

We compute the functional derivatives of $C_S(\kappa)$ from the definition (2.11 *b*)

$$\frac{\delta C_S}{\delta \chi} = 0, \quad \frac{\delta C_S}{\delta \kappa} = j'(\kappa), \quad \frac{\delta C_S}{\delta \boldsymbol{u}_b} = \boldsymbol{0}.$$

Substituting into (A 2), and using the antisymmetry of $\{\cdot,\cdot\}$, gives

$$\{C_S, G\}^{\odot} = \int_D \left[-\chi\left\{j'(\kappa), \frac{\delta G}{\delta \chi}\right\}\right] d^3x - \int_0^L \left[\frac{\delta G}{\delta w_b} rj'(\kappa)\frac{\partial \kappa}{\partial z} + \kappa \frac{\delta G}{\delta \chi}\frac{\partial}{\partial z}(j'(\kappa))\right]_{r=R} 2\pi\, dz.$$

Application of the identity (A 1) yields

$$\{C_S, G\}^{\odot} = \int_D -\frac{\delta G}{\delta \chi}\{\kappa, j'(\kappa)\}\, d^3x - \int_0^L \kappa \frac{\delta G}{\delta w_b} rj'(\kappa)\frac{\partial \kappa}{\partial z}\bigg|_{r=R} 2\pi\, dz - \int_0^{\frac{1}{2}R^2} \kappa \frac{\delta G}{\delta \chi}\frac{\partial}{\partial y}(j'(\kappa))\bigg|_0^L 2\pi\, dy. \tag{A 3}$$

Now we use the assumption that the baseflow is periodic, thus terms like

$$-\int_0^{\frac{1}{2}R^2} \kappa \frac{\delta G}{\delta \chi}\frac{\partial}{\partial y}(j'(\kappa))\bigg|_0^L 2\pi\, dy$$

vanish. Because $\delta G/\delta \boldsymbol{u}_b$ is divergence free, $\delta G/\delta w_b$ in the second term is independent of z and can be pulled out of the integral. This leaves the integral of a perfect differential in the place of the second term whose result is zero because of periodicity. Finally, $\{\kappa, j'(\kappa)\} = 0$ because κ and $j'(\kappa)$ are dependent. Thus C_S is a Casimir of the axisymmetric Lie–Poisson bracket with respect to all hamiltonian functions G including the hamiltonian (2.4). One may check that $C_S(\chi, \kappa)$ is conserved by direct differentiation of C_S with respect to time and the use of the equations of motion.

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